Additional Practice CS103 Final Exam II

Problem One: Discrete Mathematics (15 Points)

Prove, by induction on n, that for any $n \in \mathbb{N}$ with $n \geq 2$, that

$$\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \ldots \cdot \log_n (n + 1) = \log_2 (n + 1)$$

You might want to use the change of base formula for logarithms: for any $c \in \mathbb{R}$ with $c > 1$,

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Problem Two: Regular Languages (40 Points Total)

Suppose that you really, *really* don't like the string **abba** and want to build a language of everything except that string. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and consider the language $NOT_{\mathtt{abba}}$ defined as follows:

$$NOT_{\mathtt{abba}} = \{ w \in \Sigma^* \mid w \neq \mathtt{abba} \}$$

For example, $\varepsilon \in NOT_{\mathtt{abba}}$ and $\mathtt{abbabb} \in NOT_{\mathtt{abba}}$, but (unsurprisingly) $\mathtt{abba} \notin NOT_{\mathtt{abba}}$.

(i) Finite Automata (10 Points)

Design a DFA for the language $NOT_{\mathtt{abba}}$.

(ii) Regular Expressions (10 Points)

Write a regular expression for $NOT_{\mathtt{abba}}$.

(iii) Nonregular Languages (20 Points)

Let $L = \{\mathtt{0}^n\mathtt{1}^n \mid n \in \mathbb{N}\}$. Prove that no infinite subset of L is a regular language. This result shows that not only is it impossible to write a regular expression for L, but it is also impossible to write a regular expression that matches infinitely many strings from L without also matching at least one string not in L.

Problem Three: Context-Free Languages (25 Points Total)

(i) Context-Free Grammars (15 Points)

Consider the following language L over the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$:

$L = \{ w \in \Sigma^* \mid |w| \equiv_4 0$, and the first quarter of the characters in w contains at least one **b** $\}$

For example, **baaa** $\in L$, **abbbbbba** $\in L$, **bbbaaabbbaaa** $\in L$, **ababbbbbbbbb** $\in L$, but **abbb** $\notin L$, **aabbbbaa** $\notin L$, $\varepsilon \notin L$, **bbb** $\notin$ L, and **aaabbbbbbbbb** $\notin L$. Here, the first quarter of the characters in each string has been underlined.

Write a context-free grammar for L.

(ii) Pushdown Automata (10 Points)

Design a PDA for the language L from part (i) of this problem. As a reminder, L is defined as

$L = \{ w \in \Sigma^* \mid |w| \equiv_4 0$, and the first quarter of the characters in w contains at least one **b** $\}$

Problem Four: R, RE, and co-RE Languages (60 Points Total)

(i) Properties of Reductions (15 Points)

A language L is called *nontrivial* iff $L \neq \emptyset$ and $L \neq \Sigma^*$.

Prove or disprove: All nontrivial **RE** languages are mapping reducible to one another.
Prove or disprove: All nontrivial co-**RE** languages are mapping reducible to one another.

(ii) RE Languages (20 Points)

Consider the following language, which consists of all Turing machines that reject at least one string:

$$L_{R1} = \{ \langle M \rangle \mid M \text{ rejects at least one string} \}$$

Prove that $L_{R1} \in$ **RE**.

(iii) Unsolvable Problems **(25 Points)**

Prove that $L_{R1} \notin$ co-**RE**. As a reminder, L_{R1} is defined as follows:

$$L_{R1} = \{ \langle M \rangle \mid M \text{ rejects at least one string} \}$$

Problem Five: P and NP Languages (40 Points Total)

(i) Mutual Reducibility **(15 Points)**

Prove that for any language L, L is **NP**-complete iff $L \leq_P$ 3SAT and 3SAT $\leq_P L$.

(ii) Resolving P $\stackrel{?}{=}$ NP **(25 Points)**

Suppose that V is a polynomial-time verifier for an **NP**-complete language L. That is,

$$w \in L \quad \text{iff} \quad \exists x \in \Sigma^*.\ V \text{ accepts } \langle w, x \rangle$$

and

$$V \text{ runs in time polynomial in } |w|$$

Now, suppose that V is a "superverifier" with the property that for any string $w \in L$, V accepts $\langle w, x \rangle$ for almost all choices of x. Specifically, for any $w \in L$, there are at most five strings x for which V rejects $\langle w, x \rangle$.

Under these assumptions, prove that **P** = **NP**.